EXCITATION OF ELASTIC WAVES IN A HALF-SPACE CONTAINING A HORIZONTAL ELASTIC CYLINDRICAL INCLUSION

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We present a method for the rigorous analysis of wave fields excited in an elastic half-space containing a horizontal elastic cylindrical inclusion by a distributed harmonic surface load. For simplicity we illustrate the application of the method by investigating a model problem of antiplane oscillations of the composite elastic medium. We derive asymptotic representations of the solutions when the inclusion is rigidly bonded, which permit a rather simple analysis of the wave field in the medium. The proposed method can be applied without changes to similar problems in two and three dimensions. In doing this only the awkwardness of the derived relations is increased significantly.

1. We consider steady-state antiplane oscillations of an elastic half-space $X \geqslant 0$ with a density ρ and a shear modulus μ, containing an elastic cylindrical inclusion with a shear modulus μ_1 and a density ρ_1 in the region $R = \sqrt{(X-h)^2 + Y^2} \leqslant a$ $(a < h)$. The inclusion is rigidly bonded to the half-space. Shear oscillations are oriented along the generatrix of the cylinder (parallel to the Z axis). The motion of the medium is described by the elasticity theory dynamical equations in displacements (the Lamé equations), which for antiplane oscillations have the form [1] $\mu\Delta W(X, Y, t) = \rho\partial^2 W(X, Y, t)/\partial t^2$, where $W(X, Y, t)$ is the displacement of a point of the medium along the Z axis, and $\Delta = \partial^2/\partial X^2 + \partial^2/\partial Y^2$ is the Laplacian operator.

We seek the solution of the latter equation for steady-state oscillations in the form $W(X, Y, t) = w(x, y)\exp(-i\omega t)$. In this case the equation for the amplitude of the displacement takes the form

$$\Delta w(x, y) + (\rho\omega^2/\mu)w(x, y) = 0,\ x = X/a,\ Y/a = y. \tag{1.1}$$

Suppose shear stresses are prescribed on the surface of the medium

$$x = 0,\ \tau_{zx} = t(y)\,e^{-i\omega t} = \begin{cases} p(y)\,e^{-i\omega t}, & y \in [b, c], \\ 0, & y \overline{\in} [b, c]. \end{cases} \tag{1.2}$$

The displacement and stress amplitudes at infinity approach zero.

We first consider a subsidiary problem of steady-state antiplane oscillations of an elastic half-space containing a horizontal cylindrical cavity and acted upon by the harmonic load (1.2) applied to the surface of the half-space, and the load

$$\tau_{rz} = \mu\frac{\partial w}{\partial r} = Z(\varphi),\ x = H - r\cos\varphi,\ y = -r\sin\varphi,\ H = h/a \tag{1.3}$$

applied to the cylindrical cavity along the boundary $r = \sqrt{(x-H)^2 + y^2} = 1$. Here r and φ are cylindrical coordinates with the origin at the center of the cavity ($r = R/a$; $\varphi = \tan^{-1}[y/(x-H)]$).

We seek the amplitude of the displacement vector in the form

$$w(x, y) = w_1(x, y) + w_2^*(x, y). \tag{1.4}$$

Here $w_1(x, y)$ is the solution of the problem of antiplane oscillations of a uniform elastic half-space with a density ρ and a shear modulus μ under the action of a harmonic shear load $\tau_{zx} = X(y)\exp(-i\omega t)$ (generally unknown) distributed over its surface; $w_2^*(x, y) = w_2(r, \varphi)$ is the solution of the problem of antiplane oscillations of an infinite elastic space with a cylindrical cavity of radius a whose surface is loaded with the stress $\tau_{rz}|_{r=1} = Y(\varphi)\exp(-i\omega t)$:

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$$w_1(x, y) = \frac{a}{2\pi\theta\mu}\int_{\sigma}\int_{-\infty}^{+\infty} \frac{X(\eta)}{\sqrt{\alpha^2/\theta^2 - 1}} e^{-x\sqrt{\alpha^2-\theta^2}+i\alpha(\eta-y)} d\eta d\alpha, \quad (1.5)$$

$$w_2(r, \varphi) = \frac{a}{2\pi\theta\mu}\int_0^{2\pi} Y(\xi) \sum_{m=0}^{\infty} A_m \cos[m(\varphi - \eta)]\, d\eta,$$

$$A_m = H_m^{(1)}(\theta r)\Big/\left[H_{m-1}^{(1)}(\theta) - \frac{m}{\theta} H_m^{(1)}(\theta)\right].$$

Here α, η, $\theta = \rho\omega^2 a^2/\mu$; $x = X/a$; $y = Y/a$; $r = R/a$ are dimensionless parameters; the $H_\eta^{(1)}(\xi)$ are Hankel functions of the first kind [2]; the contour σ passes below the positive branch point $\alpha = +\theta$ and above the negative branch point $\alpha = -\theta$; the rest of the contour coincides with the real axis [3].

Substituting (1.5) into (1.4) and satisfying the boundary conditions on $w(x, y)$ (1.2), (1.3), we obtain the following system of integral equations for the unknown stresses $X(y)$ and $Y(\varphi)$:

$$X(y) - \frac{\varepsilon}{2\pi\theta}\int_0^{2\pi} Y(\xi) \sum_{m=0}^{\infty} \Phi_m(y, \eta)\, d\eta = t(y), \quad (1.6)$$

$$Y(\varphi) + \frac{1}{2\pi}\int_{\sigma}\int_{-\infty}^{\infty} X(\eta)\left[\cos\varphi + \frac{i\alpha\sin\varphi}{\sqrt{\alpha^2 - \theta^2}}\right] e^{-H\theta\sqrt{\alpha^2-\theta^2}(1-\varepsilon\cos\varphi)+i\alpha\sin\varphi+i\alpha\eta} d\eta d\alpha = Z(\varphi).$$

Here

$$\Phi_m(y, \eta) = \left\{\frac{H\theta\varphi_{1m}(y)}{\sqrt{y^2 + H^2}}\cos[m(\operatorname{arctg}(-y) - \eta)] - \frac{my}{y^2 + H^2}\varphi_{2m}(y)\sin[m(\operatorname{arctg}(-y) - \eta)]\right\}\Big/\Delta_m;$$

$$\varphi_{1m}(y) = H_{m-1}^{(1)}\left(\theta\sqrt{y^2 + H^2}\right) - \frac{m}{\theta\sqrt{y^2 + H^2}} H_m^{(1)}\left(\theta\sqrt{y^2 + H^2}\right);$$

$$\varphi_{2m}(y) = H_m^{(1)}\left(\theta\sqrt{y^2 + H^2}\right);\ \varepsilon = \frac{1}{H};\ \Delta_m = H_{m-1}^{(1)}(\theta) - H_m^{(1)}(\theta)\frac{m}{\theta}.$$

To solve the initial problem of the excitation of antiplane oscillations in a half-space containing an elastic cylindrical inclusion we write an expression for the wave field in an infinite elastic cylinder having a density ρ_1 and a shear modulus μ_1, and acted upon by the shear stress (1.3) distributed over the surface of the cylinder. The amplitude of the displacement of points of the elastic cylinder $w_3(r, \varphi)$ is given by the relation

$$w_3(r, \varphi) = \frac{1}{2\pi\theta_1\mu_1}\int_0^{2\pi} Z(\eta) \sum_{k=0}^{\infty} \frac{I_k(\theta_1 r)\cos[k(\varphi - \eta)]}{I_{k-1}(\theta_1) - \frac{k}{\theta_1} I_k(\theta_1)}\, d\eta, \quad (1.7)$$

where $\theta_1^2 = \rho_1\omega^2 a^2/\mu_1$, and $I_k(\zeta)$ is a modified Bessel function [2].

The rigid bonding of the cylinder to the elastic half-space ensures the continuity of displacements and stresses at the media interface. The condition of continuity of stresses is satisfied automatically by the specification of equal stresses at the interface outside and inside the inclusion. By satisfying the continuity condition for the displacements

$$w(x, y) = w_1(x, y) + w_2(r, \varphi)|_{r=1} = w_3(r, \varphi)|_{r=1}, \quad (1.8)$$
$$x = H - r\cos\varphi,\ y = -r\sin\varphi,$$

we obtain an equation which together with system (1.6) permits the determination of the functions $X(y)$, $Y(\varphi)$, and $Z(\varphi)$. It should be noted that for $h > a$ the operators in (1.6) and (1.8) are completely continuous in the space of summable functions.

After system (1.6), (1.8) has been solved, the determination of the wave field reduces to the calculation of the expressions for the amplitude functions (1.4), (1.5) outside the elastic inclusion, and (1.7) inside it.

2. We investigate the system of equations (1.6), (1.8). When h and a are commensurate ($h > a$) this system can be reduced to a quasiregular infinite system of linear algebraic equations which can be solved efficiently by numerical methods. When $h \gg a$ ($H \gg 1$, $\varepsilon \ll 1$), the system can be solved by asymptotic methods, which permits the determination of an analytic solution of the required accuracy.

Let us consider the case when the load (1.2) on the surface of the half-space is uniformly distributed over a strip, i.e. $p(y) = p_0 = \text{const}$. In this case we obtain the first approximation of the solution of the system in the form

$$X_0(y) = t(y),\ Y_0(\varphi) = Z_0(\varphi) = 0, \tag{2.1}$$

$$X(y) - X_0(y) + \varepsilon X_1(y) + \dots,\ Y(\varphi) = Y_0(\varphi) + \sqrt{\varepsilon} Y_1(\varphi) + \dots,\ Z(\varphi) = Z_0(\varphi) + \sqrt{\varepsilon} Z_1(\varphi) + \dots$$

The substitution of $X_0(y)$ from (2.1) into the second of Eqs. (1.6) gives

$$Y(\varphi) = Z(\varphi) - \sqrt{\frac{\varepsilon}{2\pi\theta}}\chi(\varphi) + O(\varepsilon), \tag{2.2}$$

where

$$\chi(\varphi) = -p_0 e^{i\frac{\pi}{4}}[G_1\cos\varphi + G_2\sin\varphi];\ G_1 = \frac{e^{\theta H\sqrt{1+c^2}}}{c(1+c^2)^{1/4}} - \frac{e^{i\theta H\sqrt{1+b^2}}}{b(1+b^2)^{1/4}};\ G_2 = \frac{e^{i\theta H\sqrt{1+c^2}}}{(1+c^2)^{1/4}} - \frac{e^{i\theta H\sqrt{1+b^2}}}{(1+b^2)^{1/4}}.$$

After substituting $X_0(y)$ into the third of Eqs. (1.8) and evaluating the integrals appearing in the expression by the method of steepest descents [4], we obtain

$$\sum_{k=0}^{\infty}[G_k\varkappa\{Z_k^{(c)}\cos k\varphi + Z_k^{(s)}\sin k\varphi\} \quad A_k\{Y_k^{(c)}\cos k\varphi + Y_k^{(s)}\sin k\varphi\}] = \sqrt{\frac{\varepsilon}{2\pi\theta}}P_0 p_0 + O(\varepsilon),\ \varkappa = \theta\mu/(\theta_1\mu_1), \tag{2.3}$$

where

$$P_0 = \left[\left\{(1+b^2)^{1/4}e^{i\frac{\theta}{\varepsilon}\sqrt{1+b^2}}\right\}\Big/ b - \left\{(1+c^2)^{1/4}e^{i\frac{\theta}{\varepsilon}\sqrt{1+c^2}}\right\}\Big/ c\right]e^{-i\frac{\pi}{4}};$$

$$\zeta_k^{(c)} = \frac{1}{2\pi}\int_0^{2\pi}\zeta(\eta)\cos k\eta d\eta;\ \zeta_k^{(s)} = \frac{1}{2\pi}\int_0^{2\pi}\zeta(\eta)\sin k\eta d\eta.$$

By expanding (2.2) in Fourier series and solving it simultaneously with (2.3), we obtain

$$Z(\varphi) = p_0\sqrt{\frac{\varepsilon}{2\pi\theta}}\left[\frac{P_0}{\varkappa G_0 - A_0} + \frac{A_1(B_1\cos\varphi + B_2\sin\varphi)}{(\varkappa G_1 - A_1)}e^{i\frac{\pi}{4}}\right] + O(\varepsilon), \tag{2.4}$$

$$Y(\varphi) = p_0\sqrt{\frac{\varepsilon}{2\pi\theta}}\left[\frac{P_0}{\varkappa G_0 - A_0} + \frac{\varkappa G_1(B_1\cos\varphi + B_2\sin\varphi)}{(\varkappa G_1 - A_1)}e^{i\frac{\pi}{4}}\right] + O(\varepsilon),$$

$$X(y) = t(y) + \frac{\varepsilon p_0}{2\pi\theta}\left[\frac{P_0 e^{i\left(\frac{\theta}{\varepsilon}\sqrt{y^2+1}+\frac{3\pi}{4}\right)}}{(y^2+1)^{3/4}(\varkappa G_0 - A_0)} + \frac{\varkappa G_1(B_1 - B_2 y)e^{i\frac{\theta}{\varepsilon}\sqrt{y^2+1}+i\frac{\pi}{4}}}{2(y^2+1)^{5/4}(\varkappa G_1 - A_1)}\right] + O(\varepsilon^{3/2}).$$

If a more accurate evaluation of $X(y)$, $Y(\varphi)$, and $Z(\varphi)$ is required, the process indicated can be continued further by taking account of higher order terms in ε in the expansion of $\Phi_m(y, \eta)$.

It should be noted that in the first approximation (2.1) we have the solution of the problem for a uniform elastic half-space. The effect of the perturbation introduced by the elastic inclusion is determined by the next term in the expansion, and is of order $\sqrt{\varepsilon}$.

To construct expressions for the wave field in the medium we substitute the values obtained for $X(y)$, $Y(\varphi)$, and $Z(\varphi)$ from (2.4) into the expressions for $w_1(x, y)$, $w_2(r, \varphi)$ (1.4), (1.5) and $w_3(r, \varphi)$ (1.7) and evaluate the integrals appearing in them by asymptotic methods [4].

Since the wave field excited by a source outside the elastic inclusion is the most informative, we present first an algorithm for calculating the functions $w_1(x, y)$ and $w_2(r, \varphi)$. Substituting the approximate value of $X(y)$ from (2.4) into the first of Eqs. (1.5) and evaluating the integrals which enter by the method of steepest descents [4], we obtain

$$w_1(x, y) = \frac{ap_0}{\mu}\sqrt{\frac{1}{2\pi\theta}}\left[\frac{\{x^2+(y+b)^2\}^{1/4}}{y+b}e^{i\theta\sqrt{x^2+(y+b)^2}} - \frac{[x^2+(y+c)^2]^{1/4}}{y+c}e^{i\theta\sqrt{x^2+(y+c)^2}}\right] + O(\varepsilon),\ c, b \sim H\ (c, b > H). \tag{2.5}$$

Similarly we have for $w_2(r, \varphi)$

$$w_2(r, \varphi) = -\frac{ap_0}{2\theta\mu}\sqrt{\frac{\varepsilon}{2\pi\theta}}\left[\frac{2P_0 H_0^{(1)}(\theta r)}{H_1^{(1)}(0)(\varkappa G_0 - A_0)} - \frac{\varkappa G_1 H_1^{(1)}(\theta r)(B_1\cos\varphi + B_2\sin\varphi)}{(\varkappa G_1 - A_1)\left(H_0^{(1)}(0) - \frac{1}{0}H_1^{(1)}(0)\right)}e^{i\frac{\pi}{4}}\right] + O(\varepsilon). \tag{2.6}$$

The amplitude w(x, y) of the displacement of the medium outside the elastic inclusion is found from (1.4) by using (2.5) and (2.6).

Since the only assumptions made in deriving (2.5) and (2.6) were that ε is small and c, $b \gg 1$, these equations are valid over the whole region, including the boundary. In (2.5) only the case y = const and $x \to \infty$ must be excluded. In this case the expression for $w_1(x, y)$ has a somewhat different form. If it is required to find the distribution of the wave field inside the elastic inclusion, we obtain by proceeding as above

$$w_3(r, \varphi) = -\frac{ap_0}{2\theta_1\mu_1}\sqrt{\frac{\varepsilon}{2\pi\theta}}\left[\frac{2P_0 I_0(\theta_1 r)}{I_0(\theta_1)(\varkappa G_0 - A_0)} - \frac{A_1 I_1(\theta_1 r)(B_1\cos\varphi + B_2\sin\varphi)}{(\varkappa G_1 - A_1)\left(I_0(\theta_1) - \frac{1}{\theta_1} I_1(\theta_1)\right)} e^{i\frac{\pi}{4}}\right] + O(\varepsilon).$$

Thus, our proposed method permits the derivation of expressions for the wave field over practically the whole region under investigation which are rather simple to analyze. It should also be noted that this method can be employed without change to treat a similar problem in two or three dimensions. In doing this only the awkwardness of the calculations is substantially increased. Thus, in treating a similar problem in two dimensions it is necessary in the first stage to solve a system of six rather than three integral equations. However, all the basic properties of the elements of the system treated in the present article are retained. In calculating wave fields the integrals and sums which arise are of the same type as in the above treatment.

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DETERMINATION OF STRESSES IN AN INFINITE PLATE WITH BROKEN OR BRANCHING CRACK

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In numerical solution of the singular integral equations which arise in two-dimensional elasticity theory problems for bodies with internally smooth curvilinear sections, the mechanical quadrature method, based on Gauss—Chebyshev quadrature expressions, is employed. Considering a piecewise-smooth crack as a limiting case of a system of smooth sections [1-3], having common points, we arrive at a system of singular integral equations with generalized singular integrands, containing fixed singularities together with the Cauchy integrand. Such equations can also be solved by the mechanical quadrature method, although more complex quadrature expressions are required (for example, Gauss—Jacoby expressions), which consider the singularity of the solution at the nodes of the section contour. Below, using the example of a broken, branching crack in an infinite plate, we present a simplified technique for numerical solution of the integral equations for piecewise-smooth sections using Gauss—Chebyshev expressions. The solution singularity at the angular point or branching point is considered inexactly, so that such a solution is only effective when it is not necessary to determine the stressed state in the vicinity of such points. In particular, the proposed solution technique will be used to determine the stress intensity coefficients at the peaks of a piecewise-smooth crack.

1. Basic Assumptions. Within an infinite plane having a related Cartesian coordinate system xOy, let there be a system of N + 1 rectilinear sections L_n, located along segments $|x_n| \leqslant l_n$ of the local coordinate axes $O_n x_n$ (n = 0, 1, ..., N). In the system xOy the origin

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